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Fixing all moduli for M-theory on K3×K3

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Abstract: We analyze M-theory compactified on $\mathrm{K3}\times\mathrm{K3}$ with fluxes preserving half the supersymmetry and its F-theory limit, which is dual to an orientifold of the type IIB string on $\mathrm{K3}\times(T^2/\mathbb{Z}_2)$. The geometry of attractive K3 surfaces plays a significant role in the analysis. We prove that the number of choices for the K3 surfaces is finite and we show how they can be completely classified. We list the possibilities in one case. We then study the instanton effects and see that they will generically fix all of the moduli. We also discuss situations where the instanton effects might not fix all the moduli.

Keywords: F-Theory, M-Theory, Superstring Vacua.

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1. Introduction

One of the simplest and most accessible forms of flux compactification is given by M-theory on K3 $\times$ K3. This was first a analyzed in [1]. The fluxes may preserve the full $N = 4$ supersymmetry, or break some or all of the supersymmetry. We will be concerned with the case where this flux breaks the supersymmetry to $N = 2$.

The F-theory limit of this theory yields an $N = 1$ theory in four dimensions and is dual, via the construction of [2], to the type IIB string on K3 $\times$ $(T^2/\mathbb{Z}_2)$, where the $\mathbb{Z}_2$ action includes an orientifolding reflection on the world-sheet.

This theory, mainly in the orientifold language, was analyzed in [3]. The fluxes themselves obstruct many of the moduli of K3 $\times$ K3 but, at least if one uses the rules of the supergravity limit described in [4], one cannot fix all of the moduli. It is believed that there are possibilities of flux obstruction beyond those found in supergravity [5], but the rules for this are not yet understood properly so we will not consider this possibility in this paper.

The fluxes select a preferred complex structure on K3 $\times$ K3 and a given choice of flux determines this complex structure uniquely. There remain up to 20 undetermined complexified Kähler moduli for each K3 surface. We will show that, in certain cases, all of these remaining moduli are generically fixed by M5-brane instanton corrections to the superpotential.

It has been realized recently [6 − 10] that fluxes may modify Witten's [11] analysis of which divisors M5-instantons may wrap to give non-trivial effects. This allows for interesting instanton effects even in simple geometries, such as a tori and K3 surfaces, where naïvely one might not expect such things.

In particular, in [9] an explicit counting of fermionic zero modes on M5 with the background (2,2) primitive flux G was performed. The generalized condition for the non-vanishing instanton corrections to the superpotential in this case requires that the new, flux dependent index of the Dirac operator equals to one, $\chi_D(G) = 1$. Here $\chi_D(G) = \chi_D - (h^{(0,2)} - n)$ where χ_D is the arithmetic genus of the divisor and $0 \leq n \leq h^{(0,2)}$ is a number of solutions of a certain constraint equation which the fermionic zero modes have to satisfy in presence of fluxes. In absence of fluxes this condition is reduced back to Witten's condition [11] that $\chi_D = 1$.

In particular, for the case of K3 $\times$ K3 4-fold and divisors of the form K3 $\times \mathbb{P}^1$ without fluxes $\chi_D = 2$ and no instanton corrections to the superpotential are possible. In presence of the background (2,2) primitive flux G it was established in [9] that $n = h^{(0,2)}$, and therefore $\chi_D(G) = 1$ and instantons corrections to the superpotential are possible. The same result for K3 $\times$ K3 was obtained in [8].

Oddly enough, we will see that if one is overzealous and tries to leave fewer than 20 Kähler moduli unfixed by the flux, the possibility arises that the instanton effects might be unable to fix some of the remaining moduli.

Our interest in the model of M-theory compactified on K3 $\times$ K3 is two-fold. First of all, this is a relatively simple model, well-understood in the framework of IIB string theory and 4d gauged supergravity [12]. The geometry of K3 surfaces is far-better understood than generic Calabi-Yau threefolds and fourfolds and so this model can be analyzed more thoroughly than the many previous examples [13 − 16] with all moduli fixed. Secondly, this model has practical applications to cosmology of D3/D7 brane inflation in type IIB string on K3 $\times$ $(T^2/\mathbb{Z}_2)$ [17, 18].

The geometry of fluxes on K3 $\times$ K3 is a very beautiful subject and has connections with number theory as analyzed in [19]. Here we will show that this allows for a complete analysis of all possibilities. In the case that the flux is purely of the type that breaks half the supersymmetries, we list all 13 possibilities that arise. Of these, only 8 correspond to orientifolds.

In section 2 we will analyze the conditions imposed on the K3 surfaces by a flux which breaks half the supersymmetry. This contains some very pretty mathematics associated to "attractive K3 surfaces". In section 3 we discuss the role of M5-brane instantons and argue that all the moduli will be generically fixed, except possibly in some cases where a particular choice of flux is made. We conclude in section 4.

2. Moduli spaces and fluxes

In this section we review the analysis of fluxes for M-theory on K3 $\times$ K3 and its F-theory limit. The latter is equivalent to an orientifold of the type IIB string on K3$\times(T^2/\mathbb{Z}_2)$. While this has been analyzed quite extensively in [3], we present a slightly different approach which more closely follows [19] which we believe is a little more efficient.

2.1 M-theory

Let us begin with M-theory on $S_1 \times S_2$, where each S_j is a K3 surface. For compactification on an 8-manifold X, an element of G-flux may be present. This G-flux is subject to a quantization condition [20], which asserts that, in our case[1]

$$G \in H^4(S_1 \times S_2, \mathbb{Z})\,. \tag{2.1}$$

A consistent theory must contain M2-branes and/or nonzero G-flux in this background satisfying [21]

$$n_{\mathrm{M2}} + \frac{1}{2}G^2 = 24\,. \tag{2.2}$$

The M2-branes will not break any supersymmetry, but the G-flux may. The supergravity analysis of [4] showed that G must be primitive and of type (2,2) in order that any supersymmetry be preserved. Any such integral 4-form may be decomposed

$$G = G_0 + G_1\,, \tag{2.3}$$

where

$$\begin{aligned} G_0 &= \sum_{\alpha=1}^{M} \omega_1^{(\alpha)} \wedge \omega_2^{(\alpha)} \\ G_1 &= \mathrm{Re}\left(\gamma \Omega_1 \wedge \overline{\Omega}_2\right)\,, \end{aligned} \tag{2.4}$$

and $\omega_j^{(\alpha)}$ are (cohomology classes[2] of) integral primitive (1,1)-forms on S_j, Ω_j is the holomorphic 2-form on S_j and γ is a complex number which must be chosen to make the last term integral.

There are essentially three kinds of moduli which arise in such a compactification:

1. Deformations of the K3 surfaces S_1 and S_2.
2. Motion of the M2-branes.
3. Deformations of vector bundles with nonabelian structure group associated to enhanced gauge symmetries arising from singular points in S_1 and S_2.

By assuming, from now on, that $n_{M2} = 0$ and that our K3 surfaces are smooth, we will restrict attention to only the first kind of modulus in this paper.

The moduli space of M-theory on K3 $\times$ K3 is of the form $\mathscr{M}_1 \times \mathscr{M}_2$, where each factor is associated to one of the K3 surfaces. If no supersymmetry is broken by fluxes, each of the $\mathscr{M}_j$ factors is a quaternionic Kähler manifold. Ignoring instanton corrections, each $\mathscr{M}_j$ is of the form

$$\mathrm{O}(\Gamma_{4,n_j})\backslash\, \mathrm{O}(4, n_j)/(\mathrm{O}(4) \times \mathrm{O}(n_j))\,, \qquad j = 1, 2\,, \tag{2.5}$$

where Γ_{4,n_j} is a lattice of signature $(4, n_j)$. The values of $n_j \leq 20$ will be determined by the choice of flux G. The space (2.5) should be viewed as the grassmannian of space-like 4-planes in $\Pi_j \subset \Gamma_{4,n_j} \otimes \mathbb{R}$ divided out by the discrete group of automorphisms of the lattice Γ_{4,n_j}.

[1] We have absorbed a factor of 2π into G compared to much of the rest of the literature.

[2] We only discuss *cohomology classes* of forms in this paper but we will usually not state this explicitly to avoid cluttering notation.

The grassmannian (2.5) is familiar (for $n_j = 20$) from the moduli space of $N = (4,4)$ superconformal field theories associated to the sigma model with a K3 target space S. In this case, the degrees of freedom parametrized by the conformal field theory are given by a Ricci flat metric on S together with a choice of $B \in H^2(S, \mathrm{U}(1))$. The choice of metric on a K3 surface of volume one is given by a space-like 3-plane $\Sigma \subset H^2(S,\mathbb{R}) = \mathbb{R}^{3,19}$. The 3-plane Σ is spanned by the real and imaginary parts of Ω, and the Kähler form J. The extra data of the B-field and volume extend this to a choice of space-like 4-plane $\Pi \subset H^*(S,\mathbb{R}) = \mathbb{R}^{4,20}$. We refer to [22] and references therein for a full account of this.

Even though M-theory itself has no B-field, the M5-brane wrapped on one K3 gives us an effective B-field for compactification on the other K3. Hence the form (2.5). We refer to [5] for examples.

A K3 surface is a hyperkähler manifold and thus has a choice of complex structures for a fixed metric. This choice corresponds to specifying the direction of J in Σ. Since supersymmetries are constructed for complex structures, this multiplicity of complex structures implies the existence of a specific extended supersymmetry.

If $G_1 = 0$ in (2.3) then the condition that G be primitive and of type (2,2) preserves the freedom to rotate Ω and J within Σ. Thus, values of G purely of the form G_0 preserve the full $N = 4$ supersymmetry in three dimensions [1].

If the term G_1 in (2.3) is non-trivial then we destroy the symmetry of rotations within Σ and the supersymmetry is broken to $N = 2$. This is the case of interest and we therefore assume, from now on, that G_1 is nonzero.

2.2 Attractive K3 surfaces

For now, let us assume that G is purely of type G_1, i.e., $G_0 = 0$. Let

$$\Omega_j = \alpha_j + i\beta_j\,. \tag{2.6}$$

for $\alpha_j, \beta_j \in H^2(S_j,\mathbb{R})$. From $\int_{S_j} \Omega_j \wedge \overline{\Omega}_j > 0$ and $\int_{S_j} \Omega_j \wedge \Omega_j = 0$, it follows that[3]

$$\begin{aligned} \alpha_j^2 &= \beta_j^2 > 0 \\ \alpha_j.\beta_j &= 0 \\ \alpha_j &\neq \beta_j\,. \end{aligned} \tag{2.7}$$

We also have

$$G = \alpha_1 \wedge \alpha_2 + \beta_1 \wedge \beta_2\,, \tag{2.8}$$

where we set $\gamma = 1$ in (2.4) by rescaling Ω_1. Let $\boldsymbol{\Omega}_j$ be a 2-plane in $H^2(S_j,\mathbb{R})$ spanned by α_j and β_j. We claim

Theorem 1. $\boldsymbol{\Omega}_1$ *and* $\boldsymbol{\Omega}_2$ *are uniquely determined by* G.

To prove this we first use the Künneth formula which tells us that

$$H^4(S_1 \times S_2,\mathbb{Z}) \cong H^0(S_1,\mathbb{Z}) \otimes H^4(S_2,\mathbb{Z}) \oplus H^2(S_1,\mathbb{Z}) \otimes H^2(S_2,\mathbb{Z}) \oplus H^4(S_1,\mathbb{Z}) \otimes H^0(S_2,\mathbb{Z}). \tag{2.9}$$

[3]We use the implicit inner product $a.b = \int_S a \wedge b$.

We know from (2.4) that G lies entirely in the second term on the right-hand side of (2.9). Let us assume we are given G, α_j, β_j solving

$$G = \alpha_1 \otimes \alpha_2 + \beta_1 \otimes \beta_2 \,. \tag{2.10}$$

Now try to find other solutions of the form

$$G = (\alpha_1 + \alpha_1') \otimes (\alpha_2 + \alpha_2') + (\beta_1 + \beta_1') \otimes (\beta_2 + \beta_2') \,. \tag{2.11}$$

It follows that

$$\alpha_1 \otimes \alpha_2' + \alpha_1' \otimes \alpha_2 + \alpha_1' \otimes \alpha_2' + \beta_1 \otimes \beta_2' + \beta_1' \otimes \beta_2 + \beta_1' \otimes \beta_2' = 0 \,. \tag{2.12}$$

Let π_1 be the projection

$$\pi_1 : H^2(S_1, \mathbb{R}) \to H^2(S_1, \mathbb{R}) / \operatorname{Span}(\alpha_1, \beta_1) \,. \tag{2.13}$$

Thus

$$\pi_1(\alpha_1') \otimes (\alpha_2 + \alpha_2') + \pi_1(\beta_1') \otimes (\beta_2 + \beta_2') = 0 \,. \tag{2.14}$$

The only solution is to put $\pi_1(\alpha_1') = \pi_1(\beta_1') = 0$, which corresponds to not rotating $\mathbf{\Omega}_1$ at all; or putting $\alpha_2 + \alpha_2'$ or $\beta_2 + \beta_2'$ equal to zero, or making $\alpha_2 + \alpha_2'$ and $\beta_2 + \beta_2'$ collinear. The latter conditions would make the new $\mathbf{\Omega}_2$, spanned by $\alpha_1 + \alpha_1'$ and $\beta_1 + \beta_1'$ violate (2.7). We may also reverse the rôles of $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ in the argument. This completes the proof of theorem 1.

The statement that $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ are fixed by G means that *the complex structures of S_1 and S_2 and uniquely determined by a choice of flux.*

The next thing we prove is

Theorem 2. *The K3 surfaces S_1 and S_2 whose complex structures are fixed by G are forced to both be attractive.*

Before we prove this, we first review the definition of an attractive[4] K3 surface. The *Picard lattice* of a K3 surface is given by the lattice $H^{1,1}(S_j) \cap H^2(S_j, \mathbb{Z})$. The *Picard number* $\rho(S_j)$ is defined as the rank of this lattice. The surface S_j is said to be *attractive* if $\rho(S_j) = 20$, the maximal value.

Let us define

$$\Upsilon_j = \Big(H^{2,0}(S_j) \oplus H^{0,2}(S_j) \Big) \cap H^2(S_j, \mathbb{Z}) \,, \tag{2.15}$$

which is the intersection of the 2-plane $\mathbf{\Omega}_j$ with the lattice $H^2(S_j, \mathbb{Z})$ in the space $H^2(S_j, \mathbb{R})$. For a generic K3 surface Υ_j will be completely trivial, but the maximal rank of Υ_j is 2. The "transcendental lattice" is defined as the orthogonal complement of the Picard lattice in $H^2(S_j, \mathbb{Z})$. If, and only if, the rank of Υ_j is 2, the transcendental lattice will coincide with Υ_j and the K3 surface S_j will be attractive. We therefore need to prove that Υ_j is rank 2.

[4]The standard mathematical term is "singular" but as this is such a singularly misleading term, we prefer to follow Moore's choice of language from [19].

Let e^j_k, $k = 1, \ldots, 22$ be an integral basis for $H^2(S_j, \mathbb{Z})$. Expanding

$$\begin{aligned}\alpha_j &= \sum_k a_{jk} e^j_k \\ \beta_j &= \sum_k b_{jk} e^j_k \\ G &= \sum_{kl} N_{kl} e^1_k \otimes e^2_l \,,\end{aligned} \tag{2.16}$$

where a_{jk} and b_{jk} are real numbers and N_{kl} are integers (since G is an *integral* 4-form). Then (2.8) becomes

$$a_{1k}a_{2l} + b_{1k}b_{2l} = N_{kl}\,, \qquad \text{for all } k, l\,. \tag{2.17}$$

Fixing l, the above equation may be read as saying that a real combination of α_1 and β_1 lies on a lattice point of $H^2(S_1, \mathbb{Z})$. By varying l we get 22 different such combinations. The fact that α_2 and β_2 are linearly independent means that all these lattice points cannot be collinear. Thus $\mathbf{\Omega}_1$ contains a 2-dimensional lattice. Similarly $\mathbf{\Omega}_2$ contains a 2-dimensional lattice and we complete the proof of theorem 2.

Attractive K3 surfaces were completely classified in [23]. They were shown to be in one-to-one correspondence with $\mathrm{SL}(2,\mathbb{Z})$-equivalence classes of positive-definite even integral binary quadratic forms. Such a quadratic form can be written in terms of a matrix

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \tag{2.18}$$

where $a, b, c \in \mathbb{Z}$, $a > 0$, $c > 0$, and $\det Q = 4ac - b^2 > 0$. Two forms Q and Q' define an equivalent K3 surface if, and only if, $Q' = M^T Q M$, for some $M \in \mathrm{SL}(2,\mathbb{Z})$.

Let Υ_j be spanned (over the integers) by integral vectors p_j and q_j. The above lattice is then

$$Q_j = \begin{pmatrix} p_j^2 & p_j . q_j \\ p_j . q_j & q_j^2 \end{pmatrix}, \tag{2.19}$$

We are free to rescale Ω_1 and Ω_2 (since they are only defined up to complex multiplication) so that

$$\Omega_j = p_j + \tau_j q_j\,, \tag{2.20}$$

for a complex number τ_j, which is fixed by the condition $\Omega_j^2 = 0$ to be

$$\tau_j = \frac{-p_j . q_j + i\sqrt{\det Q_j}}{q_j^2}\,. \tag{2.21}$$

Note that this choice of rescaling means we cannot now assume $\gamma = 1$ in (2.4). We then obtain

$$G = (\mathrm{Re}(\gamma) p_1 \otimes p_2 + \mathrm{Re}(\gamma\tau_1) q_1 \otimes p_2 + \mathrm{Re}(\gamma\bar{\tau}_2) q_2 \otimes p_1 + \mathrm{Re}(\gamma\tau_1\bar{\tau}_2) q_1 \otimes q_2)\,. \tag{2.22}$$

Consider the condition imposed by the integrality of G. Since $p_1 \otimes p_2$ is integral and primitive[5] we must have $\mathrm{Re}(\gamma) \in \mathbb{Z}$. The other terms on (2.22) put further conditions of γ. It is easy to show that a consistent choice of γ making each term in (2.22) integral is

[5] Primitive in the sense that it is not an integral multiple of a lattice element.

possible if and only if $\sqrt{\det(Q_1Q_2)}$ is an integer. That is,

Theorem 3. *A pair of attractive K3 surfaces S_1 and S_2 will correspond to a choice of integral G-flux if and only if* $\det(Q_1Q_2)$ *is a perfect square.*

Finally we need to impose the tadpole condition $\frac{1}{2}G^2 = 24$. We compute

$$G^2 = \frac{1}{4}(\gamma\Omega_1 \wedge \overline{\Omega}_2 + \bar{\gamma}\Omega_2 \wedge \overline{\Omega}_1)^2 , \tag{2.23}$$

and use the fact that $\Omega_j^2 = 0$ so only the cross term in the square is not vanishing. Therefore

$$G^2 = \frac{1}{2}|\gamma|^2 \int \Omega_1 \wedge \overline{\Omega}_1 \int \Omega_2 \wedge \overline{\Omega}_2 . \tag{2.24}$$

Using $\Omega_j = p_j + \tau_j q_j$ we find

$$\frac{1}{2}G^2 = \frac{|\gamma|^2 \det(Q_1Q_2)}{q_1^2 q_2^2} = 24 . \tag{2.25}$$

Solving (2.25) together with the integrality of (2.22) provides all possibilities of flux compactifications with $G = G_1$. We may prove that there is a finite number of attractive K3 surfaces that yield solutions to this equation as follows. We use the following theorem from [24]:

Theorem 4. *In the equivalence class of the matrix (2.18) under the action of* $\mathrm{SL}(2,\mathbb{Z})$*, assuming that* $-\det(Q)$ *is not a perfect square, one can always find a representative matrix satisfying*

$$|b| \le |c| \le |a| . \tag{2.26}$$

In our case $-\det(Q)$ is negative and so, clearly, not a perfect square. Thus we may restrict attention to matrices satisfying the above bounds. Putting

$$Q_1 = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} , \qquad Q_2 = \begin{pmatrix} 2d & e \\ e & 2f \end{pmatrix} , \tag{2.27}$$

yields

$$\det(Q_1) = 4ac - b^2 \ge 4ac - ac = 3ac . \tag{2.28}$$

Similarly, $\det(Q_2) \ge 3df$. Thus (2.25) yields

$$\frac{1}{2}G^2 \ge \frac{9}{4}|\gamma|^2 ad . \tag{2.29}$$

Suppose $\mathrm{Re}(\gamma) \ne 0$. Then $|\gamma|^2 \ge 1$, since $\mathrm{Re}(\gamma) \in \mathbb{Z}$. In this case we are done since a and d are positive integers and b, c, e, f are constrained by theorem 4. On the other hand, if γ is purely imaginary, then the integrality of the second term in (2.22) forces

$$\frac{\mathrm{Im}(\gamma)\sqrt{\det(Q_1)}}{2c} \in \mathbb{Z} . \tag{2.30}$$

Q_1	Q_2	γ	O?	Q_1	Q_2	γ	O?
$\begin{pmatrix}4&0\\0&2\end{pmatrix}$	$\begin{pmatrix}4&0\\0&2\end{pmatrix}$	$1+\frac{i}{\sqrt{2}}$	✗	$\begin{pmatrix}4&2\\2&4\end{pmatrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$2+\frac{2i}{\sqrt{3}}$	✓
$\begin{pmatrix}4&2\\2&4\end{pmatrix}$	$\begin{pmatrix}6&0\\0&2\end{pmatrix}$	$1+\frac{i}{\sqrt{3}}$	✓	$\begin{pmatrix}6&0\\0&4\end{pmatrix}$	$\begin{pmatrix}6&0\\0&4\end{pmatrix}$	$\frac{2i}{\sqrt{6}}$	✗
$\begin{pmatrix}6&0\\0&6\end{pmatrix}$	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$1+i$	✗	$\begin{pmatrix}6&0\\0&6\end{pmatrix}$	$\begin{pmatrix}4&0\\0&4\end{pmatrix}$	1	✓
$\begin{pmatrix}8&4\\4&8\end{pmatrix}$	$\begin{pmatrix}4&2\\2&4\end{pmatrix}$	$1+\frac{i}{\sqrt{3}}$	✓	$\begin{pmatrix}12&0\\0&2\end{pmatrix}$	$\begin{pmatrix}12&0\\0&2\end{pmatrix}$	$\frac{i}{\sqrt{6}}$	✗
$\begin{pmatrix}12&0\\0&4\end{pmatrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$1+\frac{i}{\sqrt{3}}$	✓	$\begin{pmatrix}12&0\\0&4\end{pmatrix}$	$\begin{pmatrix}6&0\\0&2\end{pmatrix}$	$\frac{i}{\sqrt{3}}$	✓
$\begin{pmatrix}12&0\\0&6\end{pmatrix}$	$\begin{pmatrix}4&0\\0&2\end{pmatrix}$	$\frac{i}{\sqrt{2}}$	✗	$\begin{pmatrix}12&0\\0&12\end{pmatrix}$	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	1	✓
$\begin{pmatrix}16&8\\8&16\end{pmatrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$1+\frac{i}{\sqrt{3}}$	✓				

Table 1: The 13 pairs of matrices Q_1, Q_2 yielding the possible attractive K3 surfaces. The column headed "O?" shows 8 solutions when one of K3 surfaces is a "Kummer surface".

Obviously $\mathrm{Im}(\gamma)$ cannot be zero since then $|\gamma|^2$ would be zero. We then have

$$\frac{1}{2}G^2 \geq \frac{c\det(Q_2)}{f} \geq 3dc\,. \tag{2.31}$$

This bounds c and d. Similarly we may use the third term in (2.22) to bound a and f. Thus we complete the proof that there are only a finite number of possibilities for a, b, c, d, e, f, and thus only a finite number of attractive K3 surfaces whenever G^2 is bounded.

In fact, it is not hard to perform a computer search to yield the full list of possibilities. For $\frac{1}{2}G^2 = 24$, i.e., $G_0 = 0$, there are 13 possibilities up to $\mathrm{SL}(2,\mathbb{Z})$ equivalence which we list in table 1. The column labeled "O?" will be explained in section 2.3. In principle, a given pair of attractive K3 surfaces might admit many, but finitely many, inequivalent choices of G. In our case, where the numbers are quite small, this never happens.

One may, of course, obtain other possibilities by considering a nonzero G_0. In this case, we solve the same problem for $\frac{1}{2}G^2 < 24$. As one might expect, the number of possibilities for a given $\frac{1}{2}G^2 < 24$ are somewhat fewer than above.

As stated above, the complex structures of S_1 and S_2 are fixed. What remains unfixed is the Kähler form and B-field degree of freedom. Using the assumption $G_0 = 0$ in (2.3), all 20 such complex degrees of freedom remain undetermined by the fluxes. A non-trivial choice of G_0 will fix some of these 20 remaining moduli.

The choice of G fixes a 2-plane within Π_j spanned by the real and imaginary parts of Ω_j. This means that the moduli space (2.5) is reduced to

$$\mathscr{M}_j \cong \mathrm{O}(\Gamma_{2,n_j})\backslash\,\mathrm{O}(2,n_j)/(\mathrm{O}(2)\times\mathrm{O}(n_j))\,. \tag{2.32}$$

If $G_0 = 0$ then $n_1 = n_2 = 20$. If G_0 is nonzero, these numbers will decrease.

2.3 The orientifold

Now let us turn our attention to the related question of orientifolds on $\mathrm{K3} \times (T^2/\mathbb{Z}_2)$. One obtains this orientifold via F-theory.

Begin with M-theory on $S_1 \times S_2$ (ignoring flux for now) to obtain an $N = 4$ theory in three dimensions as above. Assume that S_2 is an elliptic K3 surface with a section. Let $\pi : S_2 \to B$ denote this elliptic fibration of S_2. By shrinking the area of the elliptic fibre, one moves to an F-theory fibration corresponding to a type IIB compactification on $S_1 \times B$. This yields a four-dimensional $N = 2$ compactification.

This four-dimensional theory can be compactified on a circle thus regaining the three-dimensional theory we had originally from the M-theory compactification. The relationship between the moduli spaces of the three-dimensional theory and four-dimensional theory can be understood from this fact. The moduli space of the three-dimensional theory is $\mathscr{M}_1 \times \mathscr{M}_2$, where each $\mathscr{M}_j$ is quaternionic Kähler. The four-dimensional theory has a moduli space $\mathscr{M}_H \times \mathscr{M}_V$, where $\mathscr{M}_H$, the hypermultiplet moduli space, is exactly $\mathscr{M}_1$.

The vector multiplet moduli space $\mathscr{M}_V$, is special Kähler. The complex dimension of $\mathscr{M}_V$ is one less than the quaternionic dimension of $\mathscr{M}_2$. Quantum corrections make for a very complicated relationship between $\mathscr{M}_V$ and $\mathscr{M}_2$. Let us ignore these quantum corrections for now, which we may do since we are only making qualitative statements about the moduli space. In this case, ignoring any flux effects or M2-branes, we have, locally

$$\mathscr{M}_2 = \frac{\mathrm{O}(4,20)}{\mathrm{O}(4) \times \mathrm{O}(20)}\,, \tag{2.33}$$

and, from the c-map [25]

$$\mathscr{M}_V = \frac{\mathrm{O}(2,18)}{\mathrm{O}(2) \times \mathrm{O}(18)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\,. \tag{2.34}$$

The first factor of (2.34) corresponds to the complex structure moduli space of S_2 if we declare S_2 to be an elliptic fibration with a section. In F-theory language, this corresponds to the moduli space of the location of 7-branes. The second factor of (2.34) would naïvely correspond to the complexified area of the base, B, of the elliptic fibration as this is the only modulus remaining once the fibre is shrunk to zero size. When moving between dimensions one must be careful with taking into account overall scalings of the metric. The result is that the second factor of (2.34) actually corresponds to the complexified volume of the K3 surface S_1. The area of the base becomes a parameter in the hypermultiplet moduli space $\mathscr{M}_H$. We refer to [12] for more details.

Sen [2] showed how type IIB orientifolds could be obtained from F-theory compactifications. Elliptic fibrations may contain "bad fibres", i.e., fibres which are not elliptic curves. These bad fibres have been classified by Kodaira. We refer to [22], for example, for a review. In Sen's analysis one takes a limit in the moduli space of complex structures of the elliptic fibration such that all the bad fibres become type I_0^* in the Kodaira classification. We now have a type IIB string compactified on the orientifold $S_1 \times (C/\mathbb{Z}_2)$, where C is an elliptic curve and the base of the elliptic fibration is $B \cong C/\mathbb{Z}_2$.

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This limit freezes the location of the F-theory 7-branes making the moduli space locally

$$\mathscr{M}_V = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\,. \tag{2.35}$$

These three complex moduli can be identified as

- The modulus of the F-theory elliptic fibre — i.e., the axion-dilaton of the type IIB string.
- The modulus of the elliptic curve C, where the base of the elliptic fibration of S_2 is $B \cong C/\mathbb{Z}_2$.
- The complexified volume of S_1, as above.

The 16 moduli that we have "lost" in passing from (2.34) to (2.35) are regained by allowing D7-branes to move away from the 4 O7-planes.

Now consider the effect of flux in the form of G_1 so as to yield an $N = 1$ supersymmetric theory in four dimensions. This flux fixes the complex structure of S_1 and S_2 making both of these K3 surfaces attractive. First note that any attractive K3 surface is elliptic with a section [23] so our condition for an F-theory limit is automatically satisfied.

The flux causes the dimension of $\mathscr{M}_H \cong \mathscr{M}_1$ to be halved — exactly as it was in the case of M-theory in section 2.2. For $\mathscr{M}_V$, the first factor of (2.34) is a complex structure moduli space and so disappears completely. In orientifold language, we fix the dilaton-axion, the complex structure of $C \cong T^2$, and the location of all the D7-branes. All that remains unfixed in $\mathscr{M}_V$ is a single complex modulus corresponding to the complexified volume of S_1.

Sen's orientifold limit of F-theory is a limit of complex structure, but once we turn on flux, we have no deformations of the complex structure! The only way our M-theory compactification can correspond to an orientifold is if the elliptic fibration of the attractive S_2 has this fibration structure to begin with.

So let us suppose S_2 is an attractive K3 surface which is an elliptic fibration with only smooth, or type I_0^* fibres. The base of such a fibration must be $\mathbb{P}^1$ and there must be exactly four I_0^* fibres and no other singular fibres. In this case, the J-invariant of the fibre has no zeros or poles and is therefore a constant. This is exactly the same elliptic fibration data as one would obtain for a K3 surface which is a "Kummer surface", i.e., a blow-up of a quotient $A/\mathbb{Z}_2$, where A is a 4-torus (or abelian surface to be more precise). It follows that S_2 is indeed a Kummer surface (following, for example, [26, proposition 2.7]).

Any Kummer surface, which is attractive, must be a $\mathbb{Z}_2$-quotient of an attractive abelian surface (see, for example, [27, equation (5.8)]). Such abelian surfaces are classified in much the same way as attractive K3 surfaces, that is, they are again in one-to-one correspondence with $\mathrm{SL}(2,\mathbb{Z})$-equivalence classes of positive-definite even integral binary quadratic forms. If Q is the matrix associated with the binary quadratic form of our attractive Kummer surface S_2 and R is the matrix associated with the attractive abelian surface A, then one can show that (see [28], for example)

$$Q = 2R\,. \tag{2.36}$$

It follows that the attractive K3 surface S_2 is a Kummer surface if, and only if, the associated even binary quadratic form is twice another even binary quadratic form. *Only the F-theory compactifications on* $\mathrm{K3} \times \mathrm{K3}$ *which satisfy this property will have orientifold interpretations.*

Looking back at table 1, we see that 8 of our 13 possibilities admit an orientifold interpretation. The column headed "O?" denotes whether an orientifold model exists.

One might be concerned that one should check that G is compatible with the F-theory limit as spelt out in [29]. That is, G should have "one leg" in the fibre direction. This condition turns out to be automatically satisfied, at least in the case $G_0 = 0$, as we explain as follows.

The spectral sequence for the cohomology of a fibration yields

$$H^2(S, \mathbb{R}) = H^0(B, R^2\pi_*\mathbb{R}) \oplus H^1(B, R^1\pi_*\mathbb{R}) \oplus H^2(B, \pi_*\mathbb{R}), \tag{2.37}$$

where $H^p(B, R^q\pi_*\mathbb{R})$ may be schematically viewed as a form with p legs in the base direction and q legs in the fibre direction.

The term $H^2(B, \pi_*\mathbb{R})$ is dual to the base $B \cong \mathbb{P}^1$. The term $H^0(B, R^2\pi_*\mathbb{R})$ is dual to the fibres including components of singular fibres. Both of these terms correspond to curves in S and thus forms of type (1,1). Therefore any form of type (0,2) or (2,0) must be contained $H^1(B, R^1\pi_*\mathbb{R})$. It follows that G has one leg in the fibre direction assuming $G = G_1$ in (2.3).

Any attractive abelian surface A must be of the form $C \times C'$, where C and C' are isogenous elliptic curves admitting complex multiplication [30]. Here, "isogenous" means that C' is isomorphic, as an elliptic curve, to a free quotient of C by any finite subgroup of $\mathrm{U}(1) \times \mathrm{U}(1)$. We refer to [31, 19] and references therein for a nice account of complex multiplication.

The elliptic fibration of the Kummer surface S_2 will therefore be an elliptic fibration with base $C/\mathbb{Z}_2$ with fibre C'. It follows from Sen's argument [32] that *F-theory on* $S_1 \times S_2$ *is equivalent to the type IIB orientifold on* $S_1 \times (C/\mathbb{Z}_2)$ *where the dilaton-axion of the type IIB theory is given by the* τ*-parameter of the elliptic curve* C'.

The fixing of the complex structures of C and C' account for the removal of the first two factors of (2.35) in the vector multiplet moduli space. The fact that C and C' are isogenous means that their τ-parameters will be related by an $\mathrm{GL}(2, \mathbb{Q})$ transformation. In other words,

$$\tau_{C'} = \frac{a\tau_C + b}{c\tau_C + d}, \tag{2.38}$$

where a, b, c, d are integers *not* necessarily satisfying $ad - bc = 1$.[6]

We should note the fact that an attractive abelian surface may, in general, be decomposed into $C \times C'$ in many inequivalent ways (other than the trivial exchange of C and C'). Thus, a fixed $S_1 \times S_2$ might be associated to none, or many inequivalent orientifold limits. An algorithm for determining a complete set of such factorizations was presented

[6]There is an example in [3] which appears to violate this condition. This is because the basis defined in the appendix of [3] is not a valid integral basis for $H^2(S, \mathbb{Z})$ and so the resulting G is not actually in integral cohomology.

in [27]. For example, if the abelian surface corresponds to

$$R = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}, \tag{2.39}$$

then one may factorize into a pair of elliptic curves with $\tau_C = \omega$ and $\tau_{C'} = 6\omega$; or $\tau_C = 2\omega$ and $\tau_{C'} = 3\omega$, where $\omega = \exp(2\pi i/3)$. In our cases, listed in table 1, such an ambiguity never occurs.

3. Instanton corrections

So far we have completely ignored any quantum corrections to the moduli space. Consider first the case of M-theory on $S_1 \times S_2$ where the flux does not break any supersymmetry. This yields an $N = 4$ theory in three dimensions. By the usual counting, any instanton solution that breaks half the supersymmetry will modify the prepotential and thus deform the metric on the moduli space. These instantons will not obstruct any moduli and the dimension of the moduli space will be unchanged by these quantum corrections.

The only source of such instanton corrections in M-theory will correspond to M5-brane instantons wrapping holomorphically embedded complex 3-folds within $S_1 \times S_2$. Such divisors are clearly of the form $S_1 \times C_g$, or $C_g \times S_2$, where C_g is an algebraic curve of genus g.

Following, [11], one can show that these divisors will only contribute nontrivially to the prepotential if they have holomorphic Euler characteristic $\chi_{\mathscr{O}} = 2$. Since $\chi_{\mathscr{O}}(\mathrm{K3} \times C_g) = 2(1-g)$, we see that our instantons must be of the form $S_1 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times S_2$.

Now suppose we turn flux on so as to break half the supersymmetry. The superpotential of the resulting low-energy effective theory will now receive instanton corrections from M5-branes wrapping divisors. A naïve interpretation of [11] would lead one to believe that one would look for divisors with $\chi_{\mathscr{O}} = 1$. There are no such divisors in $S_1 \times S_2$ and so one would arrive at the conclusion that the Kähler moduli cannot be removed.

This is not the case however. It was shown in [8 – 10] that some fermion zero modes on the M5-brane worldvolume are lost changing the counting argument of [11]. The result is that, with the G-flux we are using, the desired instantons should have $\chi_{\mathscr{O}} = 2$. That is, the instantons which contribute to the superpotential are precisely those wrapping $S_1 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times S_2$.

As discussed in the previous section, the complex structure on S_1 and S_2 is completely fixed by the choice of G-flux. Each K3 surface is attractive and, as such has Picard number equal to 20. This leaves each K3 surface with 20 complexified Kähler form moduli. If G is purely of the form $G = G_1 = \mathrm{Re}(\Omega_1 \wedge \overline{\Omega}_2)$, then these 20 moduli are unfixed by the fluxes. Any terms from G_0 in (2.3) will fix some of these remaining 20 moduli.

In any case, at least in the supergravity approximation, one cannot remove all of these Kähler moduli by fluxes. It is possible to fix at least 10 of the Kähler moduli but in the F-theory limit one is restricted to fixing only 2 Kähler moduli using G_0 effects. It is conceivable that going beyond the supergravity approximation may change such statements as discussed in [5].

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Let S be an attractive K3 surface and let $V = \mathrm{Pic}(S) \otimes \mathbb{R} = \mathbb{R}^{20}$ be the subspace of $H^2(S, \mathbb{R})$ spanned by the Kähler form. We wish to find a convenient basis for V. Let us consider an element of $H^2(S, \mathbb{R})$ as a homomorphism from 2-chains in S to $\mathbb{R}$. If $\alpha \in H^2(S, \mathbb{R})$ and x is a 2-chain, we thus denote $\alpha(x) \in \mathbb{R}$.

The following proposition will be useful

Proposition 1. *We may find a set $\{e_1, \ldots, e_{20}\}$ of holomorphically embedded $\mathbb{P}^1$'s in S and a basis $\{\xi_1, \ldots, \xi_{20}\}$ of V such that $\xi_a(e_b) = \delta_{ab}$.*

To see this we use the fact that any attractive S is an elliptic fibration $\pi : S \to B$ with at least one section as noted in section 2.3. Now take any rational curve (i.e., holomorphically embedded $\mathbb{P}^1$) $C \subset S$. Since π is a holomorphic map, the image of C under π is either a point or all of B. In the former case C is a component of a singular fibre and in the latter case C is a "section" (or multisection) of the fibration.

In [33, theorem 1.1] it is shown that the complete Picard lattice of an elliptic surface with a section is generated by rational combinations of sections, smooth fibres and components of singular fibres. If there is at least one bad fibre which is reducible, the smooth elliptic fibre itself is homologous to a sum of smooth rational curves. This is indeed the case for attractive K3 surfaces as shown in [23]. The proposition then follows.

An instanton correction to the superpotential from an M5-brane wrapping a divisor D will be of the form $\sim f \exp(-\mathrm{Vol}(D))$, where $\mathrm{Vol}(D)$ is the complexified volume of D. The coefficient f may depend on complex structure moduli but cannot depend on the Kähler moduli. This is because f is computed perturbatively and the "axionic" shift symmetry of the complex partner to the Kähler form prevents any contribution to perturbation theory.

Using the bases $\{e_1^{(1)}, \ldots, e_{20}^{(1)}\}$ for $H_2(S_1)$ and $\{e_1^{(2)}, \ldots, e_{20}^{(2)}\}$ for $H_2(S_2)$ from our proposition we have volumes of the form

$$\begin{aligned} &\mathrm{Vol}(S_1)\,\mathrm{Area}(e_a^{(2)}) \\ &\mathrm{Area}(e_a^{(1)})\,\mathrm{Vol}(S_2)\,. \end{aligned} \tag{3.1}$$

The volume of S_j is determined from the Kähler form which is determined by the areas of the $\mathbb{P}^1$'s. Proposition 1 then implies we have 40 independent functions on 40 variables. If the superpotential is a suitably generic function then we therefore expect classical vacua to be isolated in the Kähler moduli space. That is, *we fix all the moduli*.

There are two known effects that can spoil the genericity of an instanton contribution and make it vanish. Firstly, the instanton may have a moduli space of vanishing Euler characteristic in some sense. This is not true in our case as rational curves in K3 surface are always isolated. The second effect can be caused by fluxes [34] as we now discuss.

3.1 Obstructed instantons

Let D be a threefold corresponding to a potential instanton $S_1 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times S_2$. Without loss of generality, we assume the instanton is of the form $C_1 \times S_2$ from now on, with $C_1 \cong \mathbb{P}^1$. Let $i : D \hookrightarrow S_1 \times S_2$ be the embedding. The term

$$\int_D b_2 \wedge i^* G\,, \tag{3.2}$$

in the M5-brane worldvolume action induces a tadpole for the anti-self-dual 2-form b_2 if $i^*G \neq 0$. We therefore demand that $i^*G = 0$ is a necessary condition for any divisor D to be considered an instanton.

How strong is the constraint $i^*G = 0$? Let us first consider the supersymmetry-breaking part of the flux $G_1 = \mathrm{Re}(\Omega_1 \wedge \overline{\Omega}_2)$. Viewing $G \in H^4(S_1 \times S_2, \mathbb{Z})$ as a homomorphism from chains in $S_1 \times S_2$ to $\mathbb{Z}$, we may write

$$i^*G(x) = G(i(x))\,, \tag{3.3}$$

where x is a 4-chain on D. Purely on dimensionality grounds, from (2.3), it is easy to see that, if $G(i(x)) \neq 0$, then x must be mapped under i to a 2-chain on S_1 and a 2-chain on S_2. We therefore suppose that $i(x) \cong C_1 \times C_2$, for some 2-cycle $C_2 \subset S_2$. But then $G(i(x)) = 0$ since Ω_1 is of type $(2,0)$ and therefore must vanish on any $\mathbb{P}^1$ (as the latter is dual to a (1,1)-form). This means that *none* of our instantons are ruled out by this part of the G-flux.

Now let us consider the case where G_0 is nonzero and given by (2.4). These fluxes will fix some of the 20 Kähler moduli. The primitivity condition for G means that J_j will be a valid Kähler form for S_j only if J_j is perpendicular all the $\omega_j^{(\alpha)}$'s.[7] Let is denote this space of Kähler forms $V_j^0 \subset H^2(S_j, \mathbb{R})$. That is,

$$V_j^0 = \bigcap_\alpha \omega_j^{(\alpha)\perp}\,, \tag{3.4}$$

where the perpendicular complement is taken with respect to $\omega_j^{(\alpha)}$ in the 20 dimensional space $\mathrm{Pic}(S_j) \otimes \mathbb{R}$.

Such a nonzero G_0 will also rule out certain instantons. Consider a 4-cycle $x \cong C_1 \times C_2$, where both C_j's are rational curves in S_j and let ξ_j denote the Poincaré dual of C_j. Then

$$G(i(x)) = \sum_\alpha (\omega_1^\alpha . \xi_1)(\omega_2^\alpha . \xi_2)\,. \tag{3.5}$$

The instanton $C_1 \times S_2$ is therefore only valid (i.e., $i^*G = 0$) if ξ_1 is orthogonal to all the $\omega_1^{(\alpha)}$'s. That is,

$$\xi_1 \in V_1^0\,. \tag{3.6}$$

Our instantons only contribute nontrivially to the superpotential if they correspond to $\mathbb{P}^1 \times \mathrm{K3}$, where we assume the $\mathbb{P}^1$ is holomorphically and smoothly embedded in the K3 surface. That is, the $\mathbb{P}^1$ is a rational curve. Fortunately these rational curves can be categorized using properties of the lattice at hand. Any rational curve in a K3 surface has self-intersection -2. This means it is Poincaré dual to an element of the lattice $H^2(S_j, \mathbb{Z})$ of length squared -2. Conversely, if ξ is an element of length squared -2 in $H^2(S_j, \mathbb{Z})$ then either ξ or $-\xi$ is Poincaré dual to a rational curve.

This leads to the following:

[7] Here we have mentioned only the real Kähler form. The complex partner of the Kähler form is similarly obstructed as discussed in [5], for example.

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Theorem 5. *If G_0 is zero we generically fix all moduli. With a nonzero G_0, instanton effects will generically fix all moduli if, and only if, the spaces V_1^0 and V_2^0 defined in (3.4) are spanned by elements corresponding to rational curves. That is the V_j^0's are spanned by elements in $V_j^0 \cap H^2(S_j, \mathbb{Z})$ of length squared -2.*

In simple cases, all the moduli are fixed. For example, suppose $M = 1$ in (2.4) and $(\omega_1^{(1)})^2 = -2$. Suppose further that that Picard lattice contains a copy of the (negated) E_8 lattice as a summand and that $\omega_1^{(1)}$ is an element of this lattice. Then the orthogonal complement of this vector will be the E_7 lattice which is generated by vectors of length squared -2.

It would be interesting to find examples where the moduli are *not* all fixed by instanton effects. This would involve analyzing sublattices in $H^2(S, \mathbb{Z})$ which are not generated by vectors of length squared -2.

3.2 The orientifold limit

By going to the F-theory limit we may obtain the equivalent statement about instanton effects in the orientifold on K3 $\times (T^2/\mathbb{Z}_2)$. Begin with M-theory on $S_1 \times S_2$, where S_2 is an elliptic fibration. Let the area of the generic elliptic fibre be A. To take the F-theory limit we set $A \to 0$.

The rescaling involved in this limit means that the volume of the M5-brane instanton must scale as A, as $A \to 0$, in order that this instanton has a nontrivial effect [11]. It follows that the instanton must either wrap an elliptic fibre, or a component of a bad fibre.

The instantons corresponding to $\mathbb{P}^1 \times S_2$ indeed wrap the fibre and so descend to D3-brane instantons wrapped around $\mathbb{P}^1 \times (T^2/\mathbb{Z}_2)$ in the F-theory limit. The instantons corresponding to $S_1 \times \mathbb{P}^1$ will be trivial unless the $\mathbb{P}^1$ corresponds to a component of a bad fibre. In this case, the D3-brane instanton becomes wrapped on $S_1 \times \mathrm{pt}$.

The moduli fixing then proceeds in the same way as it did for M-theory. Unless an inauspicious choice of G-flux is used, all the moduli should be fixed by instanton effects as follows. After flux was applied, the single remaining modulus in $\mathscr{M}_V$ corresponded to the volume of S_1. Clearly this is fixed by the instantons wrapping $S_1 \times \mathrm{pt}$. The remaining moduli correspond to the areas of rational curves in S_1 and the area of $T^2/\mathbb{Z}_2$. Given that the volume of S_1 has been fixed, we have precisely the right number of independent constraints from the $\mathbb{P}^1 \times (T^2/\mathbb{Z}_2)$ instantons to fix these latter moduli.

The fact that the single vector multiplet corresponding to the volume of S_1 is fixed was also observed in [10], where a more quantitative analysis was performed using duality.

4. Discussion

If one considers M-theory on K3 $\times$ K3 with no M2 branes and a flux chosen to break supersymmetry down to $N = 2$ in three dimensions, then the complex structures of the two K3 surfaces are fixed. To be precise, the two K3 surfaces are both attractive K3 surfaces. There remain 20 complex moduli associated to each K3 surface which vary the Kähler form.

If we leave the 40 moduli unfixed by fluxes, then we have argued that generically one would expect instanton effects to fix all 40. If flux is used to fix further moduli then we showed that there is a possibility that some moduli can remain unfixed by instanton effects.

The obvious next step should be to compute these instanton effects more explicitly and determine the values of the moduli. This might be a difficult exercise for the following reasons.

Before the flux was turned on we have an $N = 4$ supersymmetric theory in three dimensions. Corrections from M5-brane instantons will effect the metric on the moduli space. The moduli space of this theory is a product of quaternionic Kähler moduli spaces. It is a well-known difficult problem in string theory to determine the form of such quaternionic Kähler moduli spaces when there are nontrivial instanton corrections. The problem of studying M5-brane instantons corrections to the moduli space is exactly equivalent to studying worldsheet instanton corrections to the heterotic string on a K3 surface. Preliminary analysis in the latter was done in [35], for example, but few concrete results have been attained.

It should be emphasized that, even though there has been much interesting progress on quaternionic Kähler manifolds (such as [36, 37]), these results tend to rely on the assumption that there is an isometry in the moduli space related to translations in the RR directions. This views the hypermultiplet moduli space as a fibration over some special Kähler base with a toroidal fibre given by the RR moduli. It is known (see [38], for example) that when non-perturbative corrections are taken into account, this fibration must have "bad fibres". These bad fibres will break these isometries in much the same way as an elliptic K3 surface has no isometries related to translation in the fibre direction. An interesting proposal for analyzing instanton effects on the hypermultiplet moduli space was given in [10] but it appears to rely on the existence of these isometries.

Now when we turn the flux on, the M5-brane instantons contribute to a superpotential, rather than the moduli space metric. This does not mean that the metric remains uncorrected however. Now, with the decreased supersymmetry, quantum corrections to the metric are less constrained and even more difficult to determine than if they arose purely from instantons. We see, therefore, that computing the superpotential directly from instanton computations may be very difficult.

Even without this detailed knowledge, however, we have shown that one should expect a number of flux compactifications associated to M-theory on $\mathrm{K3} \times \mathrm{K3}$ (or its equivalent orientifold $\mathrm{K3} \times (T^2/\mathbb{Z}_2)$) where all the moduli are fixed by the combined action of the flux and the instanton effects.

In the context of the F-theory limit, which is equivalent to an orientifold of the type IIB string on $\mathrm{K3} \times (T^2/\mathbb{Z}_2)$ the result of this paper shows that the goal of fixing all moduli in this model is now accomplished.[8] The first part, namely fixing the moduli by fluxes, was achieved in [3, 12] and a nice summary of this work was presented in [40]. In absence

[8] In [39] the counting of fermionic zero modes on D3 brane is performed which leads to an analogous result.

of D3-branes, the 18+1 complex moduli "unfixable" by fluxes span the scalar manifold

$$\mathscr{M}^{\text{min}}_{\text{unfixed}} = \frac{\mathrm{O}(2,18)}{\mathrm{O}(2)\times\mathrm{O}(18)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\,. \tag{4.1}$$

Here the first factor includes 18 complex fields, the remnant of the $N=2$ hypermultiplets. It includes the area of $(T^2/\mathbb{Z}_2)$ and other hypermultiplets. The second factor is the remnant of the $N=2$ vector multiplet and it describes the volume of the K3 surface S_2 and its axionic partner. This case in our setting requires that both G_1 and G_0 are non-vanishing. We have to be careful therefore and comply with the conditions of the theorem 5, where it is explained that only certain choice of fluxes G_0 will allow us enough freedom (18+1 choice of proper 4-cycles) to fix by the instantons all remaining 18+1 complex moduli in (4.1).

An even simpler case, from the perspective of instantons, is when we introduce only G_1 flux (breaking $N=2$ into $N=1$ supersymmetry) will leaves us with the 20+1 complex moduli unfixed by fluxes. They span the scalar manifold

$$\mathscr{M}_{\text{unfixed}} = \frac{\mathrm{O}(2,20)}{\mathrm{O}(2)\times\mathrm{O}(18)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\,. \tag{4.2}$$

In such case we simply have 20+1 choices for the D3 instantons wrapping the 4-cycles in $\mathrm{K3}\times(T^2/\mathbb{Z}_2)$ and all unfixed by fluxes moduli are fixed by instantons.

The whole story of fixing all moduli in the M-theory version of this model, compactified on $\mathrm{K3}\times\mathrm{K3}$ is incredibly simple and elegant. In the compactified three-dimensional model there are no vectors. Therefore without fluxes, we have two 80-dimensional quaternionic Kähler spaces, one for each K3. With non-vanishing G_1 flux, each K3 becomes an attractive K3, one-half of all the moduli are fixed, but 40 in each K3 still remain moduli and need to be fixed by instantons. There are 20 proper 4-cycles in each K3 and they provide instanton corrections from M5-branes wrapped on these cycles: the moduli space is no more.

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References

[1] K. Dasgupta, G. Rajesh and S. Sethi, *M-theory, orientifolds and G-flux*, *JHEP* **08** (1999) 023 [hep-th/9908088].

[2] A. Sen, *Orientifold limit of F-theory vacua*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 92 [hep-th/9709159].

[3] P.K. Tripathy and S.P. Trivedi, *Compactification with flux on K3 and tori*, *JHEP* **03** (2003) 028 [hep-th/0301139].

[4] K. Becker and M. Becker, *M-theory on eight-manifolds*, *Nucl. Phys.* **B 477** (1996) 155 [hep-th/9605053].

[5] P.S. Aspinwall, *An analysis of fluxes by duality*, hep-th/0504036.

[6] L. Görlich, S. Kachru, P.K. Tripathy and S.P. Trivedi, *Gaugino condensation and nonperturbative superpotentials in flux compactifications*, *JHEP* **12** (2004) 074 [hep-th/0407130].

[7] P.K. Tripathy and S.P. Trivedi, *D3 brane action and fermion zero modes in presence of background flux*, *JHEP* **06** (2005) 066 [hep-th/0503072].

[8] N. Saulina, *Topological constraints on stabilized flux vacua*, *Nucl. Phys.* **B 720** (2005) 203 [hep-th/0503125].

[9] R. Kallosh, A.-K. Kashani-Poor and A. Tomasiello, *Counting fermionic zero modes on M5 with fluxes*, *JHEP* **06** (2005) 069 [hep-th/0503138].

[10] P. Berglund and P. Mayr, *Non-perturbative superpotentials in F-theory and string duality*, hep-th/0504058.

[11] E. Witten, *Non-perturbative superpotentials in string theory*, *Nucl. Phys.* **B 474** (1996) 343 [hep-th/9604030].

[12] L. Andrianopoli, R. D'Auria, S. Ferrara and M.A. Lledo, *4-d gauged supergravity analysis of type-IIB vacua on* $K3 \times T^2/\mathbb{Z}_2$, *JHEP* **03** (2003) 044 [hep-th/0302174].

[13] F. Denef, M.R. Douglas and B. Florea, *Building a better racetrack*, *JHEP* **06** (2004) 034 [hep-th/0404257].

[14] F. Denef, M.R. Douglas, B. Florea, A. Grassi and S. Kachru, *Fixing all moduli in a simple F-theory compactification*, hep-th/0503124.

[15] J.P. Derendinger, C. Kounnas, P.M. Petropoulos and F. Zwirner, *Fluxes and gaugings: $N = 1$ effective superpotentials*, *Fortschr. Phys.* **53** (2005) 926 [hep-th/0503229].

[16] O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, *Type IIA moduli stabilization*, *JHEP* **07** (2005) 066 [hep-th/0505160].

[17] K. Dasgupta, C. Herdeiro, S. Hirano and R. Kallosh, *D3/D7 inflationary model and M-theory*, *Phys. Rev.* **D 65** (2002) 126002 [hep-th/0203019].

[18] F. Koyama, Y. Tachikawa and T. Watari, *Supergravity analysis of hybrid inflation model from D3-D7 system*, *Phys. Rev.* **D 69** (2004) 106001 [hep-th/0311191].

[19] G.W. Moore, *Les Houches lectures on strings and arithmetic*, hep-th/0401049.

[20] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[21] S. Sethi, C. Vafa and E. Witten, *Constraints on low-dimensional string compactifications*, *Nucl. Phys.* **B 480** (1996) 213 [hep-th/9606122].

[22] P.S. Aspinwall, *K3 surfaces and string duality*, in *Fields, strings and duality, TASI 1996*, C. Efthimiou and B. Greene eds., World Scientific, 1997, p. 421, hep-th/9611137.

[23] T. Shioda and H. Inose, *On singular K3 surfaces*, in *Complex analysis and algebraic geometry*, W.L. Baily and T. Shioda eds., Cambridge, 1977, p. 119.

[24] H.L. Keng, *Introduction to number theory*, Springer-Verlag, 1982.

[25] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475.

[26] R. Miranda, *The moduli of Weierstrass fibrations*, *Math. Ann.* **255** (1981) 379.

[27] T. Shioda and N. Mitani, *Singular abelian surfaces and binary quadratic forms*, in *Classification of algebraic varieties and compact complex manifolds*, Lecture Notes in Mathematics **412**, Springer-Verlag, 1974., p. 259.

[28] W. Barth, C. Peters, and A. van de Ven, *Compact complex surfaces*, Springer, 1984.

[29] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070].

[30] D. Mumford, *Abelian varieties*, Tata-Oxford, 1970.

[31] S. Gukov and C. Vafa, *Rational conformal field theories and complex multiplication*, *Commun. Math. Phys.* **246** (2004) 181 [hep-th/0203213].

[32] A. Sen, *F-theory and orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150].

[33] T. Shioda, *On elliptic modular surfaces*, *J. Math Soc. Japan* **24** (1972) 20.

[34] D. Robbins and S. Sethi, *A barren landscape*, *Phys. Rev.* **D 71** (2005) 046008 [hep-th/0405011].

[35] E. Witten, *World-sheet corrections via D-instantons*, *JHEP* **02** (2000) 030 [hep-th/9907041].

[36] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type-II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys.* **B 332** (1990) 317.

[37] R. D'Auria, S. Ferrara, M. Trigiante and S. Vaula, *Gauging the Heisenberg algebra of special quaternionic manifolds*, *Phys. Lett.* **B 610** (2005) 147 [hep-th/0410290].

[38] P.S. Aspinwall and R.Y. Donagi, *The heterotic string, the tangent bundle and derived categories*, *Adv. Theor. Math. Phys.* **2** (1998) 1041 [hep-th/9806094].

[39] E. Bergshoeff, R. Kallosh, A.K. Kashani-Poor, D. Sorokin, and A. Tomasiello, to appear.

[40] R. D'Auria, S. Ferrara and M. Trigiante, *No-scale supergravity from higher dimensions*, *Comptes Rendus Physique* **6** (2005) 199 [hep-th/0409184].

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